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# *Geometrical and Invariantive Theory of Quartic Curves Modulo 2.*[*]

By L. E. Dickson.

§ 1. The bitangents and singular points of a quartic curve $Q$ modulo 2 depend only upon the terms of $Q$ in which at least one exponent is odd. We shall write $Q=E+O$, where

$$E=ax^4+by^4+cz^4+jx^2y^2+kx^2z^2+ly^2z^2,$$
$$O=dx^3y+exy^3+fx^3z+gxz^3+hy^3z+iyz^3+mx^2yz+nxy^2z+pxyz^2.$$

The *derived*[†] points are the points for which the three first partial derivatives of $Q$ all vanish; they depend upon $O$, but not upon $E$.

Again, the bitangents depend upon $O$ alone. For example, $z=0$ is a bitangent if and only if $d\equiv e\equiv 0$ (mod 2), so that the terms free of $z$ in $Q$ form a perfect square modulo 2. The line $x=ry+sz$ is a bitangent if the quartic in $y$ and $z$ obtained by eliminating $x$ has no terms in $y^3z$ and $yz^3$. An equivalent definition is that $l=0$ is a bitangent if and only if $Q$ can be expressed in the form $lC+q^2$ (mod 2).

Under a linear transformation, $Q$ becomes $E_1+O_1$ modulo 2, in which the terms $O_1$ involving at least one odd exponent are derived solely from the similar part $O$ of $Q$.

Thus $Q$ is dominated by its $O$ in several respects, just as is the case with the terms of the second degree in the equation of a conic in non-homogeneous coördinates.

§ 2. Theorem. *Any two bitangents intersect in a derived point.*

After applying a linear transformation, we may assume that $x=0$ and $y=0$ are bitangents. Then $h=i=0$, $f=g=0$, so that $O$ is quadratic in $z$. Thus no partial derivative of $Q$ has a term $z^3$, so that each partial derivative is zero for $x=y=0$. Hence the intersection of the two bitangents is a derived point.

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[†] There is at least one, since Euler's theorem shows that the derivatives are dependent, the sum of their products by $x$, $y$, $z$ being identically zero modulo 2.

§ 3. THEOREM. *Unless all of the bitangents concur, every derived point is the intersection of two bitangents.*

By hypothesis there are three bitangents forming a triangle. They may be transformed linearly into $x=0$, $y=0$, $z=0$. Then $O$ becomes

$$O'=mx^2yz+nxy^2z+pxyz^2.$$

For the case $m=n=p=0$, $Q=E$ is a double conic and every point of the plane is a derived point and every line a bitangent.

The derived points of $E+O'$ are those for which

$$ny^2z+pyz^2=0, \quad mx^2z+pxz^2=0, \quad mx^2y+nxy^2=0.$$

The derived points having $z=0$ are evidently the intersections of the bitangent $z=0$ with one of the bitangents $x=0$, $y=0$, $mx+ny+pz=0$ (these being factors of $O'$). Next, a derived point with $z=1$ is $(s\,r\,1)$, where

$$nr^2+pr=0, \quad ms^2+ps=0.$$

Then $y=rz$ and $x=sz$ are bitangents meeting at $(s\,r\,1)$.

In view of the last two theorems, the derived points coincide completely with the intersections of the bitangents, provided the last are not all concurrent. But if there are two or more bitangents to a real quartic and all concur at $P$, there is a derived point besides $P$. For (§ 4) $O$ can then be transformed into $O_4$ or $O_6$. The bitangents to $E+O_4$ are the infinitude of lines through $P=(001)$, while any point with $x=0$ is a derived point. The bitangents to $O_6$ are $x=0$ and $y=0$, while $P$ and $(010)$ are the derived points. Finally, a real quartic with a single bitangent can be transformed into $E+O_{11}$ (§ 4), whose single derived point is $P$ and single bitangent is $y=0$.

## *Non-equivalent Systems of Real Quartics*, §§ 4–8.

§ 4. Henceforth we consider only quartics $Q$ with integral coefficients and subject them to linear transformations with integral coefficients only. The $2^6$ quartics $E+O$ in which the six coefficients of $E$ take independently the values 0 and 1, while $O$ is fixed, shall be said to form a *system.*

We shall prove that every such system is equivalent to one and but one of the fourteen systems $E+O_i$, in which $O_0$ is identically zero, while $O_1, \ldots,$ $O_{13}$ are given in the table together with all of their bitangents (where $x$ denotes $x=0$), in which

$$\rho^2+\rho+1=0, \quad A^3+A^2+1=0, \quad b^2+\rho^2b+1=0.$$

| | | |
|---|---|---|
| $O_1$ | $x^2yz+xy^2z+xyz^2$ | all 7 real lines |
| $O_2$ | $xy^2z+xyz^2$ | $x,\ y,\ z,\ y=z$ |
| $O_3$ | $x^2yz$ | $x,\ y,\ z=\alpha y$ ($\alpha$ any) |
| $O_4$ | $x^3y$ | $x,\ y=\alpha x$ ($\alpha$ any) |
| $O_5$ | $x^3y+O_1$ | $x,\ y,\ y=x,\ z=\rho x,\ z=\rho x+y$ |
| $O_6$ | $x^3y+xy^2z$ | $x,\ y$ |
| $O_7$ | $x^3y+x^2yz+xyz^2$ | $x,\ y,\ z=\rho x$ |
| $O_8$ | $x^3y+yz^3+xyz^2$ | $y,\ z=Ax$ |
| $O_9$ | $xy^2z+O_8$ | $y,\ z=Ax,\ z=Ax+A^2y$ |
| $O_{10}$ | $x^3y+y^3z+yz^3+xy^2z$ | $y,\ z=x+\rho y,\ z=\rho x+by$ |
| $O_{11}$ | $x^3y+y^3z$ | $y$ |
| $O_{12}$ | $x^3y+xz^3+y^3z+yz^3$ | $x=(s^3+1)y+sz,\ s^7+s+1=0$ |
| $O_{13}$ | $x^3y+xz^3+y^3z+xy^2z$ | $x=\sigma^3y+\sigma z,\ \sigma^7+\sigma^3+1=0$ |

§ 5. First, let the quartic have three real bitangents forming a triangle. Applying a real transformation, we obtain $O=O'$ of § 3. We obtain at once $O_0$, $O_1$, $O_2$ or $O_3$.

Second, let there be three intersecting real bitangents $x=0$, $y=0$, $x=y$, but no real ones forming a triangle. Then

$$O=dx^3y+exy^3+O', \quad m=n.$$

If $m+p=1$, $z=dx+ey$ is a bitangent, contrary to hypothesis. Hence $m=p$. Since $O\neq O'$, we may interchange $x$ and $y$, if necessary, and set $d=1$. Replacing $x$ by $x+ey$, we have $d=1$, $e=0$, and thus obtain $O_4$ or $O_5$.

Third, let there be exactly two real bitangents $x=0$, $y=0$. Then

$$O=dx^3y+exy^3+O', \quad m\neq n.$$

Replace $z$ by $z+\alpha x+\beta y$, where $\alpha$ and $\beta$ are integers. We get

$$(d+m\alpha+p\alpha)x^3y+(e+n\beta+p\beta)xy^3+O'+E.$$

Since $m+p$ and $n+p$ are not both zero, we may take $e=0$ in $O$. Then $d=1$, since $z=0$ is not a bitangent. If $p=n$, then $m+p+1=0$ and $x=z$ is a bitangent. Hence $p=n+1=m$. According as $m=0$ or 1, we have $O_6$ or $O_7$.

§ 6. Fourth, let $y=0$ be the only real bitangent. Then $f=g=0$ in $O$. Since no one of the six remaining real lines is a bitangent, the two numbers of each pair

$$d,\ e; \quad h,\ i; \quad m+n+h,\ i; \quad d+i+m+p,\ h+e;$$
$$d,\ e+n+p; \quad d+i+m+p,\ e+h+i+n+p$$

are not both zero. Suppose that $d=0$, so that $e=1$, $n=p$, by the first and fifth pairs. If $i=0$, then $h=1$, $m=n$, by the second and third pairs, contrary to the fourth. Thus $i=1$ and by interchanging $x$ and $z$ we obtain an $O$ with $d=1$, $f=g=0$. Hence we may set $d=1$. Replacing $x$ by $x+ey$, we have $e=0$. Then replacing $x$ by $x+mz$, we have also $m=0$.

Thus $d=1$, $e=m=0$, while the two numbers of each pair

$$h,\ i;\quad n+h,\ i;\quad 1+i+p,\ h;\quad 1+i+p,\ h+i+n+p$$

are not both zero. For $h=0$, we have $i=p=1$; according as $n=0$ or 1, we get $O_8$ or $O_9$. For $h=n=1$, we have $i=1$; according as $p=0$ or 1, we get $O_{10}$ or $O_{10}+xyz^2$, the latter being derived, apart from terms $E$, from $O_9$ by replacing $z$ by $z+y$. For $h=1$, $n=0$, we have $i=p$ and hence $O_{11}$ or $O''=O_{11}+yz^3+xyz^2$. The only bitangents to $O''$ are $y=0$, $z=Ax+(A+1)y$, which are derived from those to $O_8$ by replacing $x$ by $x+y$ and $z$ by $z+y$; the same replacement carries $O_8$ to $O''+E$.

§ 7. Finally, consider quartics without real bitangents. We may take $d=1$, make $f=0$ by replacing $y$ by $y+fz$, and make $e=m=0$ as in § 6. Hence

$$O=x^3y+Gxz^3+Hy^3z+Iyz^3+Nxy^2z+Pxyz^2.$$

Then $G=1$, since $y=0$ is not a bitangent. The only* bitangents are $x=ry+sz$, where

$$r=s^3+Ps+I,\quad r^2s+Nr+H=0.$$

These have the solutions $r$, $s=0, 0;\ 1, 0;\ 0, 1;\ 1, 1$ if

$$I=H=0;\quad I=1,\ H=N;\quad I=P+1,\ H=0;\quad I=P,\ H=N+1,$$

respectively. No one of the latter pairs of equations is to hold, since there is to be no real bitangent. Hence $H=1$ and either $N=1, I=0$ or $N=0$, $I=P+1$. If $N=P=0$, then $I=1$ and $O=O_{12}$. If $N=1$, $I=P=0$, $O=O_{13}$. If $N=I=0$, $P=1$, we apply $(xzy)$ and get $O_{13}$. If $N=P=1$, $I=0$, we apply $(xyz)$ and then replace $x$ by $x+z$, obtaining $O_{12}+z^4$.

§ 8. The only case in which two of our $O_j$ have the same number of real bitangents and the same number of imaginary bitangents involving the roots of irreducible congruences modulo 2 of the same degree is that of $O_{12}$ and $O_{13}$. Hence it remains only to prove that $O_{12}$ is not equivalent to any $E+O_{13}$ under a real linear transformation.

Let $B_s$ and $\beta_\sigma$ denote the bitangents to $O_{12}$ and $O_{13}$ given in the table of § 4. Since the congruence for $s$ is irreducible, the square of any root is a root. Now $B_s$ becomes $B_{s^2}$ under the transformation

$$T:\qquad x=Y+Z,\quad y=Z,\quad z=X,$$

which therefore permutes the seven bitangents $B_{s^i}$ ($i=1, 2, 4, 8, 16, 32, 64$)

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* Dickson, *Trans. Amer. Math. Soc.*, April, 1915.

cyclically; it replaces $O_{12}$ by $O_{12}+y^2z^2+z^4$. Hence, if $O_{12}$ and $O_{13}$ are equivalent, there is a real linear transformation which replaces any chosen bitangent to $O_{12}$ by any chosen one to $O_{13}$. We take $\sigma=s^4+s^5$, which is a root* of $\sigma^7+\sigma^3+1\equiv0$ (mod 2). The transformation $(c_{ij})$ replaces $B_s$ by $lX+mY+nZ=0$, where

$$l=c_{11}+(s^3+1)c_{21}+sc_{31}, \quad n=c_{13}+(s^3+1)c_{23}+sc_{33}.$$

If the resulting line is $\beta_\sigma$, we have $n=l\sigma$. Now

$$l\sigma=l(s^4+s^5)=c_{31}s^6+(c_{31}+c_{21}+c_{11})s^5+(c_{21}+c_{11})s^4+c_{21}s^2+c_{21}.$$

If this equals the above cubic function $n$ of $s$ we have, in view of the linear independence of $s^6, \ldots, s, 1$ modulo 2, $c_{31}=c_{21}=c_{11}=0$, whereas the elements of the first column of $(c_{ij})$ are not all zero.

We have now completed the proof that *every system is equivalent to one and but one of fourteen systems* $E+O_i$ $(i=0, 1, \ldots, 13)$.

## *Invariants Distinguishing between the Systems.*

§ 9. We readily construct a rational integral function $c_j$ of the coefficients of $Q$ which is congruent to zero modulo 2 if and only if the real line $l_j=0$ is a bitangent to $Q$. For example, if $l_j=x$, we may set $c_j=hi+h+i$, whose vanishing modulo 2 implies $h=i=0$, and hence that $x=0$ is a bitangent.

Let $s_1, \ldots, s_7$ denote the elementary symmetric functions of $c_1, \ldots, c_7$. Since any real linear transformation permutes the lines $l_j=0$ and hence the $c$'s, each $s_i$ is an invariant of $Q$. Now $s_i$ is unity for a quartic with exactly $7-i$ real bitangents, since exactly $i$ of the $c$'s are unity for such a quartic, while the remaining $c$'s are zero. Moreover, $s_i$ is zero for a quartic with more than $7-i$ real bitangents. Hence the values of $s_1, \ldots, s_7$ determine the number of real bitangents.

We can construct in a similar way invariants which determine the number of bitangents whose coefficients are in the Galois field of the $2^n$ polynomials in a root of a congruence of degree $n$ irreducible modulo 2. We have only to employ in place of the above seven real lines $l_i=0$ the lines with coefficients in the present field.

The values of all these invariants with $n\leqq4$ distinguish between two systems of quartics not having the same number of bitangents with coefficients in each of the fields of order $2^n$ $(n=1, 2, 3, 4)$, and hence differentiate all the systems $E+O_j$ $(j=0, \ldots, 11)$ from each other and from the two with $j=12, 13$. A new invariant is needed to differentiate the last two systems from each other.

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* Found by use of the table by Bussey, *Bull. Amer. Math. Soc.*, Vol. XII (1905–6), p. 33. The headings $i$-$\lambda$ are misprints for $i^\lambda$.

We proceed to construct an invariant $J$ whose value is 1 for any quartic equivalent to an $E+O_{12}$ and 0 for the remaining quartics, so that $J$ is a characteristic invariant for the system $E+O_{12}$. By applying the transformations specified at the beginning of § 7 to a quartic with $d=1$, we obtain $E+O$, where $O$ is of the form in § 7, with coefficients

$$G=g+f(e+p+n), \qquad H=h+e(m+n+f),$$
$$I=i+fh+Ge+pm+efm, \quad N=n+ef, \quad P=p+m+ef.$$

By §§ 7, 8, $E+O$ is equivalent to $E+O_{12}$ if and only if $G=H=1$, $N=P=I+1$, and hence if and only if $GH(N+P+1)(N+I)=1$. The case $d=0$, $e=1$ is reduced to the preceding by interchanging $x$ and $y$. Finally, if $d=e=0$, $z=0$ is a real bitangent, whence $J=0$. Thus

$$J=dGH\{(n+p+m+1)(i+fh+e+pm+efm)+NP\}$$
$$+(d+1)ef(i+ph+mh)\{(n+p+m+1)(g+fh+pn)+m(n+p)\}.$$

In view of its construction, $J$ is an invariant of the general quartic modulo 2. As a check this was verified for the substitution $(xy)$.

### *Real Linear Automorphs of Each System.*

§10. To test the equivalence of two quartics of the same system $E+O_j$, we require the real linear transformations which leave the system unaltered, *i. e.*, its automorphs.

For $j=0$ or 1, every real linear transformation is an automorph.

For $j=2$ or 3, the real bitangents are $x=0$ and the three lines through (100). Since that point and the line $x=0$ are invariant, the transformation is a binary one on $y$, $z$. Every such real binary transformation is an automorph.

An automorph of $E+O_4$ must leave fixed the intersection (001) of all the bitangents and must permute the derived points $(0kl)$, $k$, $l$ arbitrary. The resulting real transformations

$$x=X, \quad y=rX+Y, \quad z=dX+eY+Z \tag{1}$$

are evidently automorphs. The same result holds for $O_5=O_4+O_1$, since $E+O_1$ is invariant under all real linear transformations.

For $j=6$ or 7, the only real bitangents are $x=0$, $y=0$; the only real derived points are (001) and (010). Hence each of the latter must be fixed and the transformation is the identity or $z=X+Z$.

For $j=8$ or 9, the only real bitangent is $y=0$; the only real point on an imaginary bitangent (or the only real derived point) is $P=(010)$. Hence an automorph is binary on $x$, $z$. It must permute the three bitangents $z=Ax$ through $P$. We get

$$t: \quad x=X+Z, \quad y=Y, \quad z=X, \tag{2}$$

and $t^2$, $t^3=I$. Conversely, these are automorphs.

An automorph of $E+O_{10}$ leaves fixed the only real bitangent $y=0$ and the only real point (101) on an imaginary bitangent. Of the resulting transformations, the only automorphs are

$$x=rX+sY+(r+1)Z, \quad y=Y, \quad z=(r+1)X+(r+s+1)Y+rZ.$$

These are the four powers of

$$x=Z, \quad y=Y, \quad z=X+Y. \tag{3}$$

An automorph of $E+O_{11}$ leaves fixed the only bitangent $y=0$ and the only derived point (001). The actual automorphs are

$$x=X+sY, \quad y=Y, \quad z=sX+mY+Z. \tag{4}$$

Among the automorphs of $E=O_{12}$ are the seven powers of $T$ in § 8, which permute the seven bitangents cyclically. There are no other automorphs, since the identity transformation alone leaves $B_s$ unaltered. In fact, one condition is $n=ls$, in the notations of § 8, whence $c_{21}$, $c_{23}$, $c_{31}$, $c_{13}$ are zero. Thus the transformation is $x=X+cY$, $y=Y$, $z=kY+Z$. It leaves $B_s$ fixed if and only if $c=sk$, whence $c=k=0$.

The only real linear transformation replacing the bitangent $\beta_\sigma$ to $O_{13}$ by $\beta_{\sigma^2}$ is seen similarly to be

$$x=Y, \quad y=Y+Z, \quad z=X, \tag{5}$$

which replaces $O_{13}$ by $O_{13}+Y^4$. Hence the only automorphs of the system $E+O_{13}$ are the seven powers of (5).

§ 11. The number $N$ of systems equivalent to a given one $E+O_i$ is found by dividing the order 168 of the group of all real linear transformations modulo 2 by the number of automorphs of the system. Using the results in § 10, we get $N=1$ for $O_0$ or $O_1$; $N=28$ for $O_2$, $O_3$; $N=21$ for $O_4$, $O_5$; $N=84$ for $O_6$, $O_7$; $N=56$ for $O_8$, $O_9$; $N=42$ for $O_{10}$, $O_{11}$; $N=24$ for $O_{12}$, $O_{13}$. The sum of these $N$'s, each in duplicate, is $512=2^9$. This count of the total number of systems is in accord with the fact that $O$ has 9 coefficients each with 2 values. We thus have a complete check upon §§ 4–10.

## *Invariantive Classification of the Quartics in a System.*

§ 12. Let a real linear automorph $T$ of the system $E+O_j$ replace $E+O_j$ by $E'+O_j$. A polynomial $I(a, \ldots, l)$ in the arbitrary integral coefficients of $E$ shall be called an invariant of the system $E+O_j$ if, when $a', \ldots, l'$ are replaced by their expressions in $a, \dots, l$, $I(a', \dots, l')$ becomes congruent modulo 2 to $I(a, \dots, l)$ for all integral values of $a, \dots, l$ and for every linear automorph $T$ of the system.

We shall obtain a fundamental set of invariants of each system and apply them to effect a complete classification of the quartics in each system. These invariants of the systems, together with the invariants characterizing the systems (§ 9), enable us to find (§28) a fundamental set of invariants of $Q$.

For the system having $O$ identically zero modulo 2, $E$ is identically congruent to the square of a quadratic form $q$. The invariants (and covariants) of $q$ modulo 2 have been determined.* If $q$ has no real linear factor, it is equivalent to $x^2+yz$ or $y^2+yz+z^2$.

§ 13. For the system $E+O_1$ every real linear transformation is an automorph. All such transformations are generated by

$$(xy): \quad (ab)(kl), \qquad (xz): \quad (ac)(jl),$$

$$x'=x+z: \quad c'=c+a+k, \quad l'=l+j+1. \tag{$\alpha$}$$

Evident invariants are therefore $jkl$ and $P=jklabc$.

First, let both of these invariants be unity. We get $E_1+O_1$,

$$E_1=x^4+y^4+z^4+x^2y^2+x^2z^2+y^2z^2.$$

It follows that $E_1+O_1$ is invariant under all real linear transformations. Thus it has no real linear factor (true also of $E_2+O_1$).

Second, let $jkl=1$, $P=0$. After an interchange of two variables, we have $a=0$, $j=k=1$. In view of $\alpha$ and

$$x'=x+y: \quad b'=b+a+j, \quad l'=l+k+1, \tag{$\beta$}$$

we may set also $b=c=0$; we get $E_2+O_1$,

$$E_2=x^2y^2+x^2z^2+y^2z^2.$$

Third, let $jkl=0$. We may set $j=0$. By use of $\alpha$ and

$$y'=y+z: \quad c'=c+b+l, \quad k'=k+j+1, \tag{$\gamma$}$$

we may set also $l=k=0$. If $a$, $b$, $c$ are all zero, $E$ is identically zero. In the contrary case we may set $a=1$ and make $c=0$ by use of the product $\alpha\beta$. We get $E_i+O_1$, where

$$E_3\equiv0, \quad E_4=x^4, \quad E_5=x^4+y^4.$$

No two of these quartics $E_i+O_1$ are equivalent, since the first, $O_1$, is a product of four real linear factors, the second has the single real linear factor $x$, while the third has none (or since they contain 6, 4, 2 real points respectively).

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* Dickson, "Madison Colloquium," 1914, p. 76 (references, p. 70).

We readily construct invariants of the system $E+O_1$ which differentiate the last three types. We need the coefficients of the equivalent form having $j=k=l=0$.

If $j=0$, we apply $\alpha^l\gamma^k$ and get

$$ax^4+by^4+[c+(a+k)l+bk]z^4+O_1.$$

If $j=1$, $k=0$, we apply $\beta^l(yz)\gamma$ and get

$$ax^4+cy^4+[b+c+l(a+1)]z^4+O_1.$$

If $j=k=1$, $l=0$, we apply $(xz)\alpha\gamma$ and get

$$cx^4+by^4+(a+b+c+1)z^4+O_1.$$

A function having the value unity when the quartic is equivalent to one with $E_3\equiv 0$ and the value zero in the contrary case, is seen by an inspection of the preceding results to be

$$(1+j)\{(a+1)(b+1)(c+1+kl)\}+j(k+1)(a+1)(c+1)(b+1+l)$$
$$+jk(l+1)(c+1)(b+1)a.$$

It may be written in the symmetrical form

$$B=(1+jkl)(a+1)(b+1)(c+1)+(j+1)kl(a+1)(b+1)$$
$$+(k+1)jl(a+1)(c+1)+(l+1)jk(b+1)(c+1).$$

To find an invariant $C$ of the system having the value 1 or 0 according as $E+O_1$ is or is not equivalent to $x^4+O_1$, we note that it follows from the third case above that $C=a+b+c$ when $j=k=l=0$. Hence,

$$C=(j+1)[a+b+c+(a+k)l+bk]+j(k+1)[a+b+l(a+1)]$$
$$+jk(l+1)(a+1)$$
$$=a(l+1)+b(k+1)+c(j+1)+jkl+jk+jl+kl.$$

Although the invariance of $B$ and $C$ follows from their construction and the non-equivalence of the $E_i+O_1$ $(i=3, 4, 5)$, this was verified by use of $(xy)$, $(xz)$, $\alpha$.

§ 14. Each system $E+O_i$ $(i=2, 3)$ is invariant only under the group $G$ of the real binary linear transformations on $y$, $z$. We have

$$E+O_2\equiv ax^4+x^2\lambda^2+x\pi+q^2 \pmod 2,$$
$$\lambda=jy+kz, \quad \pi=yz(y+z), \quad q=by^2+lyz+cz^2.$$

Evidently $\pi$ is invariant under $G$. Hence the invariants of $E+O_2$ under $G$ are the polynomials in $a$ and the invariants of $\lambda$ and $q$ under $G$. Among the latter are evidently

$$P=(j+1)(k+1), \quad l, \quad bcl, \quad I=(l+1)(b+1)(c+1), \quad R=ljk+bk+cj,$$

$R$ being the resultant of $\lambda$ and $q$.

First, let $P=0$. We may set $j=1, k=0$, since

$$(yz): \qquad (bc)(jk), \qquad (6)$$
$$y=Y+Z: \qquad c'=c+b+l, \quad k'=k+j. \qquad (7)$$

A linear combination of our last four invariants gives $b(l+c+1)$. Hence $b$ is fixed if $l=c$, a relation between our invariants $l$ and $R=c$. But if $l=c+1$, $z=Y+Z$ adds unity to $b$ without otherwise changing $E+O_2$, so that we may set $b=0$. The respective types are

$$ax^4+by^4+cz^4+x^2y^2+cy^2z^2+O_2, \quad ax^4+cz^4+x^2y^2+(c+1)y^2z^2+O_2.$$

Second, let $P=1$, whence $j=k=0$. If $l=I=0$, $b$ and $c$ are not both zero, and we may take $b=1$, $c=0$, in view of (6), (7). If $l=0$, $I=1$, then $b=c=0$. If $l=1$, either $bcl=1$ or we may set $b=1$, $c=0$. The respective types are

$$ax^4+y^4+O_2, \quad ax^4+O_2, \quad ax^4+y^4+cz^4+y^2z^2+O_2.$$

Hence our invariants form a fundamental set. Aside from a possible factor $x$ or $y$ (which may be determined by inspection), there is no real linear factor except* $z$ and $y+z$ of $ax^4+O_2$ for $a=0$ and $y+z$ of the final type for $a=c=0$. Hence there are 11 types without real linear factors.

§ 15. The automorphs of the system $E+O_3$ form the binary group $G$ on $y$, $z$. Using $q$ of § 14, we have

$$E+O_3 \equiv ax^4+x^2\beta+q^2 \pmod 2, \quad \beta=jy^2+yz+kz^2.$$

Hence the invariants of $E+O_3$ are the polynomials in $a$ and the invariants of $\beta$ and $q$ under $G$, the latter† being polynomials in

$$l, \quad jk, \quad bcl, \quad I, \quad \sigma=l(j+k)+bk+cj+b+c,$$

$I$ as in § 14. Since we desire the types, it is about as simple to proceed independently as to make use of the result cited that $l, \ldots, \sigma$ form a fundamental set of invariants of $\beta$ and $q$.

If $l=0$, $jk=1$, either $I=1$, whence $b=c=0$ and $E+O_3$ has the factor $x^2$, or $I=0$, when we may set $b=1$, $c=0$ by use of (6) and

$$y=Y+Z: \qquad c'=c+b+l, \quad k'=k+j+1. \qquad (8)$$

If $l=jk=1$, either $b=c=1$ or $bcl=0$; in the latter case, we may set $b=0$, $c=0$ by use of (6), (8).

If $jk=0$, we may set $j=0$, $k=0$ by (6), (8). Then $\sigma=b+c$. First, let $bcl=0$, $l=1$. If $\sigma=0$, then $b=c=0$. If $\sigma=1$, we may set $b=1$, $c=0$ by (6). Second, let $l=0$. Then $I$ and $\sigma$ give $bc$ and $b+c$, so that $b$ and $c$ can be at most interchanged and this is done by (6). We may drop the case $b=0$, $c=1$.

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* The four possibilities are the four real bitangents.

† Dickson, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXI, pp. 126, 127.

Hence our invariants characterize the resulting types and thus form a fundamental set. The thirteen types without real linear factors are

$$\begin{array}{lll} ax^4+y^4+x^2y^2+x^2z^2+x^2yz, & \phi=ax^4+x^2y^2+x^2z^2+y^2z^2+x^2yz, & \phi+y^4+z^4, \\ ax^4+y^4+z^4+y^2z^2+x^2yz, & \psi=x^4+y^2z^2+x^2yz, & \psi+y^4, \\ x^4+y^4+x^2yz, & ax^4+y^4+z^4+x^2yz. & \end{array}$$

§ 16. The further systems are treated by new methods, illustrated in this section and § 18. For the system $E+O_{13}$ the seven automorphs are (5) and its powers. Under (5), $O_{13}$ receives the increment $Y^4$ and $E+O_{13}$ becomes $E'+O_{13}$, in which

$$a'=c, \quad b'=b+a+j+1, \quad c'=b, \quad j'=k+l, \quad k'=l, \quad l'=j. \tag{5'}$$

Evident invariants of the system are

$$I=(j+1)(k+1)(l+1), \quad P=abcI.$$

If $I=1$, whence $j=k=l=0$, either $P=1$, whence $a=b=c=1$, or $P=0$, giving seven sets $a$, $b$, $c$, not all unity. These sets are permuted cyclically by (5′), so that we may take $a=b=c=0$.

If $I=0$, the seven sets $j$, $k$, $l$, not all zero, are permuted cyclically by (5′), so that we may take $j=1$, $k=l=0$. The identity is the only automorph preserving these values. Hence we need invariants to fix $a$, $b$, $c$. The sum of $aj$ and its six conjugates under the powers of (5′) is the invariant

$$a(j+k+l)+b(j+l)+c(k+j)+(k+1)(j+l).$$

Similarly from $ak$ and $al$ we obtain two invariants which, when combined linearly with the preceding one, give

$$\alpha=aj+bl+c(k+l), \quad \beta=ak+b(j+k)+cl+jl+j+l,$$
$$\gamma=al+bk+cj+jk+lk+jl.$$

For our form, $j=1$, $k=l=0$, whence $\alpha=a$, $\beta=b+1$, $\gamma=c$ fix $a$, $b$, $c$. Thus $I$, $P$, $\alpha$, $\beta$, $\gamma$ *form a fundamental set of invariants of the system* $E+O_{13}$, *and the ten types of quartics are*

$$x^4+y^4+z^4+O_{13}, \quad O_{13}, \quad ax^4+by^4+cz^4+x^2y^2+O_{13}.$$

No one of these and no quartic $E+O_{12}$ has a real linear factor, having no real bitangent.

§ 17. The treatment of $E+O_{12}$ is similar to the last. Employing the automorph $T$ of § 8, we have

$$T': \quad a'=c, \quad b'=a, \quad c'=a+b+j+1, \quad j'=k, \quad k'=k+l, \quad l'=j+1.$$

Evident invariants are

$$M=jk(l+1), \quad \pi=(a+1)(b+1)(c+1)M.$$

If $\pi=1$, then $j=k=1$, $l=a=b=c=0$. If $\pi=0, M=1$, then $j=k=1, l=0$, and the seven sets $a, b, c$, not all zero, are permuted cyclically by $T'$, so that we may set $a=1, b=c=0$. If $M=0$, the seven sets $j, k, l$ are permuted by $T'$ and we may set $j=k=l=0$. Then invariants fixing $a, b, c$ are

$$al+b(k+1)+c(j+1)+l(j+1)(k+1), \quad a(j+k)+b(j+1)+cl+k+l,$$
$$a(j+l+1)+bl+c(j+k)+kl(j+1).$$

Hence the last three and $M$, $\pi$ give a fundamental set of invariants of the system. The 10 types are

$$x^2y^2+x^2z^2+y^2z^2+O_{12}, \quad x^4+x^2y^2+x^2z^2+O_{12}, \quad ax^4+by^4+cz^4+O_{12}.$$

§ 18. For $E+O_{10}$ the automorphs are the powers of (3). Under it,

$$T: \quad a'=c, \quad b'=b+c+l, \quad c'=a, \quad j'=l+1, \quad k'=k, \quad l'=k+j.$$

We readily obtain the invariants

$$k, \quad k(l+j), \quad kl(j+1), \quad \sigma=a(j+l+1)+c(k+j+l),$$
$$ac, \quad a+c, \quad \tau=(k+1)\{aj+b+cl+l(j+1)\}.$$

If $k=0$, $T$ permutes $(j, l)=(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ cyclically. Hence we may set $j=l=0$. Then $a, b, c$ are fixed by $\sigma=a$, $\tau=b$ and $a+c$.

Let $k=1$, $k(l+j)=0$. By use of $T$, we may set $l=j=0$. Now $T^2$ changes only $b$, adding $a+c+1$ to it; hence if $a+c=0$, we may set $b=0$; the invariant $ac$ then reduces to $a$. For $a+c=1$, invariants reducing to $a$ and $b$ when $k=1$, $l=j=0$, are

$$\lambda=k(j+1)(l+1)a+kljc,$$
$$\mu=k(j+1)(l+1)(a+c)b+klj(a+c)(b+c+1),$$

which were formed by starting with the first term (the desired one) and adding the term derived from it by $T$. Since $T$ carries the second term to the first, we have an invariant.

Finally, let $k=1$, $k(l+j)=1$. The invariant $kl(j+1)$ then becomes $l$. Thus $l$ and $j$ are invariant. If $a+c=1$, we apply $T$ and set $a=1$, $c=0$; then make $b=0$ by $T^2$. Next, let $a+c=0$. If $a=c=l+1$, we make $b=0$ by $T$. If $a=c=l$, $b$ is fixed by the new invariant

$$\pi=k(j+l)(a+c+1)(a+l+1)b.$$

The one real bitangent $y=0$ is a factor only when $k=a=c=0$.

Our ten invariants form a fundamental set. The nineteen types of quartics without real linear factors are

$$ax^4+by^4+cz^4+O_{10} \ (a, c \text{ not both } 0), \quad ax^4+az^4+x^2z^2+O_{10},$$
$$ax^4+by^4+(a+1)z^4+x^2z^2+O_{10}, \quad x^4+\phi,$$
$$(l+1)(x^4+z^4)+\phi, \quad l(x^4+z^4)+by^4+\phi,$$

where

$$\phi=(l+1)x^2y^2+x^2z^2+ly^2z^2+O_{10}.$$

§ 19. The only automorphs of the system $E+O_6$ are $z=X+Z$ and its square. It replaces $E+O_6$ by $E'+O_6$, in which the altered coefficients are

$$a'=a+c+k, \quad j'=j+l+1.$$

Evident invariants are $b$, $c$, $k$, $l$, $lj$ and

$$\alpha=(l+1)(a+cj+jk), \quad \beta=la(c+k+1).$$

If $l=0$, we may make $j=0$, whence $a=\alpha$. If $l=c+k=1$, we may make $a=0$, whence $j=lj$. If $l=1$, $c=k$, we have $j=lj$, $a=\beta$. Hence the invariants form a fundamental set. The twenty-one types without real linear factors are

$$ax^4+by^4+z^4+kx^2y^2+O_6, \quad ax^4+y^4+kx^2z^2+O_6 \ (a=1 \text{ or } k=1),$$
$$by^4+cz^4+jx^2y^2+(c+1)x^2z^2+y^2z^2+O_6,$$
$$ax^4+by^4+cz^4+jx^2y^2+cx^2z^2+y^2z^2+O_6 \ (a=1 \text{ or } c=1).$$

§ 20. For $E+O_7$, $a'=a+c+k$, $j'=j+l$. The invariants are

$$b, \quad c, \quad k, \quad l, \quad j(l+1), \quad al+clj+jk, \quad a(l+1)(c+k+1).$$

If $l=1$, we make $j=0$. If $l=0$, $c+k=1$, we make $a=0$. The thirty types without real linear factors are

$$ax^4+by^4+cz^4+kx^2z^2+y^2z^2+O_7 \ (\text{except } a=c=k=0),$$
$$by^4+cz^4+jx^2y^2+(c+1)x^2z^2+O_7 \ (\text{except } b=c=0),$$
$$ax^4+by^4+z^4+jx^2y^2+x^2z^2+O_7, \quad x^4+y^4+jx^2y^2+O_7.$$

§ 21. For $E+O_8$ the automorphs are the three powers of $t$ given by (2). Now $t$ replaces $E+O_8$ by $E'+O_8$, in which

$$t: \quad a'=a+c+k, \quad c'=a, \quad j'=j+l, \quad l'=j.$$

A fundamental set of invariants is given by

$$b, \quad k, \quad kac, \quad P=(k+1)(a+1)(c+1), \quad \pi=(l+1)(j+1),$$
$$\rho=al+cj+jkl, \quad \sigma=aj+c(j+l)+k(l+1).$$

If $\pi=1$, then $l=j=0$. If also $P=1$, then $k=a=c=0$ and $y$ is a factor. But if $P=0$, $k=0$, then $a$, $c$ are not both zero and by use of $t$ and its powers we may set $a=c=1$. If $k=1$, either $ac=1$ or we may make $a=c=0$ by use of $t$ and its powers.

If $\pi=0$, we may set $l=1$, $j=0$ by use of $t$ and its powers. Then $a=\rho$, $c=\sigma$, $b$ and $k$ are fixed by invariants.

The twenty types without a real linear factor are

$$ax^4+by^4+az^4+kx^2z^2+O_8, \quad ax^4+by^4+cz^4+kx^2z^2+y^2z^2+O_8,$$

with $a$, $c$, $k$ not all zero.

§ 22. The automorphs of $E+O_9$ are the powers of $t$ given by (2), while

$$t: \quad a'=a+c+k, \quad c'=a, \quad j'=j+l+1, \quad l'=j.$$

A fundamental set of invariants is given by

$$b,\quad k,\quad lj,\quad \lambda=kac,\quad \mu=(k+1)(a+1)(c+1),$$
$$\sigma=a(j+1)+c(j+l)+kl(j+1),\quad \tau=a(j+l)+c(l+1)+jk(l+1).$$

Let $lj=1$, whence $l=j=1$. According as $k=1$ or $k=0$ the powers of $t$ permute the three sets $a$, $c$ not all 1, or not all 0, respectively, so that we may set $a=c$. Then $\lambda$, $\mu$ give the value of $a$.

If $lj=0$ we may take $l=j=0$ by use of powers of $t$. Then $\sigma=a$, $\tau=c$.

The twenty types without a real linear factor are

$$ax^4+by^4+az^4+x^2y^2+kx^2z^2+y^2z^2+O_9 \text{ (except } a=k=0),$$
$$ax^4+by^4+cz^4+kx^2z^2+O_9 \text{ (except } a=c=k=0).$$

§ 23. The four automorphs of $E+O_{11}$ are given by (4). We get

$$a'=a+s(c+k),\quad b'=b+sa+mc+sj+msk+ml+m+s,$$
$$l'=l+sk,\quad j'=j+(m+s)k+sl+s.$$

A fundamental set of invariants is given by

$$c,\quad k,\quad l(k+1),\quad lj(k+1),\quad A=a(c+k+1),\quad B=(k+1)(l+1)(a+cj),$$
$$C=c(k+1)(l+1)(b+ja),\quad D=l(k+1)(c+1)(a+j)b,$$
$$E=k(c+1)(l+a),\quad F=kla+kb+k(l+j)c+kj(l+1).$$

First, let $k=0$, $l(k+1)=0$, whence $l=0$. We make $j=0$ by choice of $s$. Then $a=B$, $cb=C$. If $c=1$, $b$ is fixed. If $c=0$, we make $b=0$ by taking $s=0$, $m=b$.

Second, let $k=0$, $l(k+1)=1$, whence $l=1$. Then $j$ is fixed by $lj(k+1)$. If $c=1$, we make $a=0$, $b=0$. If $c=0$, $a=A$, $(a+j)b=D$, so that $b$ is fixed if $a+j=1$. But if $a=j$, we can make $b=0$.

Third, let $k=1$. We can make $l=j=0$. Then $A=ac$, $E=a(c+1)$ fix $a$, while $F=b$.

The eighteen types with no real linear factor are

$$ax^4+by^4+z^4+O_{11},\quad x^4+O_{11},\quad z^4+jx^2y^2+y^2z^2+O_{11},\quad x^4+by^4+y^2z^2+O_{11},$$
$$x^4+x^2y^2+y^2z^2+O_{11},\quad ax^4+by^4+cz^4+x^2z^2+O_{11}.$$

§ 24. The automorphs of $E+O_4$ are given by (1). We have

$$a'=a+br+cd+jr+kd+lrd+r,\quad b'=b+(c+l)e,$$
$$j'=j+ke+lre+ld,\quad k'=k+lr.$$

A fundamental set of invariants is given by

$$c,\quad l,\quad P=(k+1)(l+1),\quad \beta=(b+1)(c+l+1),\ \pi=P(j+1),$$
$$\lambda=P(c+1)(b+j)a,\quad \mu=l(c+1)[a+k(b+j+1)],$$
$$\tau=ck(l+1)(b+j),\quad \tau a,\quad \sigma=lc(a+bk+jk+j+k).$$

First, let $l=1$, $c=0$. We may make $b=k=j=0$. Then $\mu=a$.

Second, let $l=c=1$. We make $k=j=0$. Then $\beta=b+1$, $\sigma=a$.

Third, let $l=0$, $c=1$. We make $b=0$ and take $e=0$ henceforth. Then $k$ is fixed by $P$ and $j$ is fixed by $\pi$ if $k=0$, while $j=\tau$ if $k=1$. If $j=k=1$, $\tau a=a$. But if $j$, $k$ are not both 1, we may set $a=0$, since

$$a'=a+d(1+k)+r(1+j)$$

Fourth, let $l=c=0$. Then $k$ and $b$ are fixed by $P$ and $\beta$. If $P=1$, $k=0$ and $j=\pi+1$. Then $a=\lambda$ if $b+j=1$; while if $b+j=0$, we make $a'=a+r=0$. Finally, let $P=0$, whence $k=1$. We make $j'=j+e=0$. Then, for $e=r=0$, $a'=a+d$ can be made zero.

The twelve types with no real linear factor are

$$x^4+y^2z^2+x^3y, \quad ax^4+by^4+z^4+y^2z^2+x^3y,$$
$$ax^4+z^4+x^2y^2+x^2z^2+x^3y, \quad z^4+jx^2y^2+kx^2z^2+x^3y \ (jk=0),$$
$$x^4+y^4+x^3y,^* \quad y^4+x^2z^2+x^3y.$$

§ 25. For $E+O_5$, we must add to the expressions for $a'$, $j'$, $k'$ in § 24 $rd$, $d+e+re$, $r$, respectively. A fundamental set of invariants is

$$c, \quad l, \quad lk, \quad lckj, \quad \xi=b(c+l+1), \quad \eta=lck(b+j)a,$$
$$\zeta=c(l+1)[a+b(k+1)+jk+j+k], \quad \lambda=(l+1)(c+1)[a+k(b+j+1)],$$
$$\mu=l(c+1)[j+b(k+1)], \quad \nu=l(c+1)(k+1)(b+j)a.$$

If $l=0$, $c=1$, we make $b=k=j=0$. Then $\zeta=a$.

If $l=c=0$, we make $k=j=0$. Then $\xi=b$, $\lambda=a$.

If $l=1$, $c=0$, we make $b=0$ and take $e=0$ henceforth. Then $lk=k$, $\mu=j$. Since $a'=a+r(j+1)+kd$, we may make $a'=0$ unless $j=1$, $k=0$. In the latter case, $\nu=a$.

If $l=c=1$, $\xi=b$, $lk=k$, while

$$j'=j+(k+1)e, \quad a'=a+r(b+j+1)+(k+1)d.$$

For $k=0$, we can make $j=a=0$. Finally, let $k=1$. Then $lckj=j$. If $b=j$, we can make $a'=0$. But if $b+j=1$, $\eta=a$.

The fourteen types with no real linear factor are

$$ax^4+z^4+O_5, \quad x^4+y^4+O_5, \quad x^4+x^2y^2+y^2z^2+O_5,$$
$$jx^2y^2+x^2z^2+y^2z^2+O_5, \quad by^4+z^4+y^2z^2+O_5,$$
$$by^4+z^4+bx^2y^2+x^2z^2+y^2z^2+O_5, \quad ax^4+by^4+z^4+(b+1)x^2y^2+x^2z^2+y^2z^2+O_5.$$

## *Rejection of Reducible Types.*

§ 26. By §§ 12–25, there are 203 non-equivalent types of quartics with no real linear factor. Of these, eight have no real point, one has seven real points, and six have six real points.†

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* To this binary form can be reduced any quartic equivalent to a binary one and having no real linear factor and not a perfect square.

† *Trans. Amer. Math. Soc.*, April, 1915.

In order to obtain a list of types of irreducible quartics, it remains to delete those types which are products $BC$ of quadratic forms without real linear factors. As noted at the end of § 12, such a quadratic form is equivalent to $x^2+yz$ or $(y+\rho z)(y+\rho^2 z)$, where $\rho^2+\rho+1=0$.

Consider quartics $Q$ with an imaginary linear factor $f$ whose coefficients involve the single imaginary $\rho$. Since $f$ is a bitangent to $Q$, the table in § 4 shows that, in $Q=E+O_i$, $i$ is 3, 4, 5, 7, or 10, and gives the possible forms of $f$. For example, if $i=3$, then $f=z+\rho y$, which is a factor of $E+O_3$ if and only if $a=0$, $b=l=c$, $j=k=1$. If $b=0$, the factor $x$ occurs. If $b=1$, we have

$$y^4+z^4+x^2y^2+x^2z^2+y^2z^2+x^2yz;$$

viz., $\phi+y^4+z^4(a=0)$, at the end of § 15. An $E+O_4$ with the factor $y+\rho x$ has the factor $x$ or $y$. An $E+O_5$ with either of the factors $z+\rho x$, $z+\rho x+y$ has both (if without the factor $y$) and is

$$x^4+z^4+x^2y^2+x^2z^2+y^2z^2+O_5=(x^2+xz+z^2)(x^2+y^2+z^2+xy+xz);$$

it is the final quartic in § 25 for $a=1$, $b=0$.

An $E+O_7$ with the factor $z+\rho x$, but not $x$, is

$$x^4+z^4+x^2z^2+O_7;$$

it is the first of the two at the end of § 20 with $a=1$, $b=j=0$.

An $E+O_{10}$ with the factor $z+x+\rho y$ is

$$Q_a=ax^4+(a+1)(y^4+z^4)+ax^2y^2+x^2z^2+ay^2z^2+O_{10}.$$

Now $Q_1$ becomes $Q_0$ under the transformation $T$ of § 18; while $Q_0$ is the third one of the list at the end of § 18 for $a=0$, $b=1$.

§ 27. Finally, let $Q$ be the product $BC$ of two quadratic forms without real or imaginary linear factors. As noted, we may take

$$B=x^2+yz,\quad C=a_1yz+a_2xz+a_3xy+b_1x^2+b_2y^2+b_3z^2,\quad \Delta=a_1a_2a_3+\Sigma a_ib_i=1.$$

First, let $a_2$ and $a_3$ be not both zero. Interchanging $y$ and $z$, if necessary, we may set $a_3=1$. The automorph

$$x=X+Z,\quad y=Y+Z,\quad z=Z \tag{9}$$

of $B$ replaces $C$ by $C'$, in which $a_3'=1$, $a_2'=a_2+1$. Hence we may set $a_3=1$, $a_2=0$. Using $\Delta=1$, we get

$$C=ayz+xy+bx^2+b_2y^2+(ab+1)z^2.$$

Let $T$ be an automorph of $B$ which replaces $C$ by a form $C'$ differing from $C$ only by terms in $x^2$, $y^2$, $z^2$. The single derived point (apex) of $B$ is $P=(100)$;

the apex of $C$ is $A=(a01)$. The three real points not on $B$ and distinct from $P$ lie on the covariant line $x+y+z=0$. The only real transformation, leaving fixed this line as well as $P$ and $A$, is seen at once to be the identity or

$$x=X+Y,\ y=Y,\ z=Y+Z.$$

It replaces $C$ by a form in which only $b_2$ is changed, the increment to $b_2$ being $a+b+ab$. The latter can be made unity unless $a=b=0$. Hence in $C$ we may set $b_2=0$ unless $b_2=1$, $a=b=0$.

Second, let $a_2=a_3=0$. By $\Delta=1$, $a_1=b_1=1$. Hence $C=B+l^2$, where $l=b_2y+b_3z$. Unless $l\equiv 0$, we may set $b_2=1$ and make $b_3=0$ by use of (9).

Hence $C$ can be transformed by an automorph of $B$ into one and but one of the seven forms

$$B,\quad B+y^2,\quad e=xy+y^2+z^2,\quad f=ayz+xy+bx^2+(ab+1)z^2.$$

The desired quartics are therefore equivalent to products of the latter by $B$. The product $B^2$ occurs at the end of § 12. To $B(B+y^2)$ we apply $(xzy)$ and get the third type (for $a=0$) at the end of § 24. Next, $Be=x^2y^2+x^2z^2+O_{10}$; it is the last type in § 18 for $l=b=0$. For $a=b=1$, $Bf=x^4+y^2z^2+O_6$; it is the last type in § 19 for $a=1, b=c=j=0$. For $a=b=0$, we replace $x$ by $x+z$ in $Bf$ and get $z^4+x^2z^2+y^2z^2+O_5$; it is next to the last type at the end of § 25 for $b=0$. In $Bf$ with $a=b+1$ we replace $x$ by $x+z$ and get

$$bx^4+(b+1)z^4+by^2z^2+x^2z^2+O_9.$$

If $b=1$, we apply transformation (2) and get $z^4+x^2z^2+O_9$, which is the preceding for $b=0$. This type is a case of the last one in § 22.

Our lists in §§ 12–25 therefore contain $5+6$ quartics (including the two in § 12) which are reducible. *There remain* 192 *non-equivalent types** *of irreducible quartics modulo* 2.

### *Fundamental Set of Invariants of the Quartic Q.*

§ 28. Evident linear combinations of the invariants of $Q$ constructed in § 9 give characteristic invariants of the various systems; $C_j$ is characteristic for the system $E+O_j$ provided $C_j=1$ if and only if $Q$ is equivalent to a quartic of this system. For example, invariant $J$ of § 9 is characteristic for the system $E+O_{12}$.

For each $j$ we constructed in §§ 12–25 a fundamental set $S_j$ of invariants $I_j$ of $E+O_j$, in which the coefficients of $E$ are arbitrary integers, and determined type forms $E_1+O_j$, $E_2+O_j$, .... to which all such forms are equivalent.

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* The only one equivalent to a binary form is that in the last line of § 24.

Let the coefficients of $Q=E+O$ be arbitrary integers subject only to the condition $C_j=1$. Compute by §§ 4–7 the coefficients $a', \ldots, l'$ of $E'$ in the form $E'+O_j$ into which $Q$ can be transformed. Let $K_j$ be the function of the coefficients of $Q$ which is obtained from $I_j(a', \ldots, l')$ by giving to $a', \ldots, l'$ their values in terms of the coefficients of $Q$. Then if $I_j$ ranges over the set $S_j$ and $j$ ranges over the values $0, 1, \ldots, 13$, the $C_j$ and the products $C_j K_j$ are invariants of $Q$ forming a fundamental set.

To prove this, let $I_{t,j}$ be the linear combination of the $I_j$ which gives that invariant of $E+O_j$ which is characteristic of the class of quartics $E+O_j$ which are equivalent to $E_t+O_j$; thus $I_{t,j}$ has the value 1 if and only if $E+O_j$ is equivalent to $E_t+O_j$. Let $K_{t,j}$ be the corresponding $K_j$. Then $C_j K_{t,j}$ has the value 1 if and only if $Q$ is equivalent to $E_t+O_j$ (the value 0 if not equivalent to it); hence it is an invariant of $Q$ and is characteristic of the class of quartics equivalent to $E_t+O_j$. Since the various $CK$ completely characterize the classes of quartics, they form a fundamental set of invariants of $Q$.